Unit 2 Matrices - Section 2i Invariant Points and Invariant Lines

Invariant points

When a point or set of points undergoes a transformation, an invariant point is one that does not change its position.

For example, in any reflection, points actually on the mirror line do not move and so would be invariant points. In this case, there would be a whole set of points that would have to be defined by an equation which would be the equation of the mirror line.

For a rotation, only one point remains in the place it started. That would be the point at the centre of rotation.

To find invariant points

This is quite straightforward. If a transformation is applied to any point \((x, y)\), which would be written as the matrix \(
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\), the point would remain unchanged so the matrix multiplication would give the result \(
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\) too.

Examples

1. Find the invariant points under the transformation given by the matrix \(
\begin{pmatrix}
    -1 & 2 \\
    1 & 0
\end{pmatrix}
\).

The point \((x, y)\) would map onto itself so \(
\begin{pmatrix}
    -1 & 2 \\
    1 & 0
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix} = \begin{pmatrix}
    x \\
    y
\end{pmatrix}.
\)

Multiplying this out gives \(\begin{pmatrix}
-x + 2y \\
x
\end{pmatrix} = \begin{pmatrix}
x \\
y
\end{pmatrix}.
\)

We have two equations \(\left\{
\begin{array}{l}
-x + 2y = x \\
x = y
\end{array}\right.
\) which simplify to \(\begin{array}{l}
x = y \\
x = y
\end{array}\) since both equations lead to the same line, \(y = x\), there is a line of invariant points lying along that line. We could write the set of points using a parameter e.g. \((\lambda, \lambda)\).

If there is only one solution, the two equations found from the matrix multiplication will not simplify to the same thing. This means that the solution to the equations would give the one invariant point under the transformation.
2. Find the invariant points under the transformation given by the matrix \(
\begin{pmatrix}
3 & 4 \\
1 & 2
\end{pmatrix}
\). 

\[
\begin{pmatrix}
3 & 4 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

\[
\begin{pmatrix}
3x + 4y \\
x + 2y
\end{pmatrix}
=
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

Equations: \(\begin{cases}
3x + 4y = x \\
3x + 2y = y
\end{cases}\)

which simplify to \(\begin{cases}
x = -2y \\
x = -\frac{1}{3}y
\end{cases}\)

The two equations can only be true if \(x = 0\) and \(y = 0\) so the invariant point is \((0,0)\).

3. Find the invariant points under the transformation given by the matrix \(
\begin{pmatrix}
4 & 1 \\
6 & 3
\end{pmatrix}
\).

\[
\begin{pmatrix}
4 & 1 \\
6 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

\[
\begin{pmatrix}
4x + y \\
6x + 3y
\end{pmatrix}
=
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

Equations: \(\begin{cases}
4x + y = x \\
6x + 3y = y
\end{cases}\)

which simplify to \(\begin{cases}
y = -3x \\
y = -3x
\end{cases}\)

The invariant points would lie on the line \(y = -3x\) and be of the form \((\lambda, -3\lambda)\).

**Invariant lines**

A line is an invariant line under a transformation if the image of a point on the line is also on the line. This is simplest to see with reflection.

If you look at the diagram on the next page, you will see that any line that is at 90° to the mirror line is an invariant line.
Examples

1. Enlargement, centre (0,0)

Any line passing through the origin is an invariant line.

We would say that any line of the form $y = kx$ is an invariant line.
2. Rotation of $180^\circ$ about (0,0)

Again, any line through (0,0) is an invariant line.

Just as before, we would say that any line of the form $y = kx$ is an invariant line.

3. Reflection in the line $y = x$

In this case, any line at $90^\circ$ to the mirror line is an invariant line.

One way of describing these lines is as lines of the form $y = -x + c$

This looks better if you write $x + y = c$ instead.